J. Fluid Mech. (1982), *vol.* 125, *pp.* 375–377
Printed in Great Britain

Note on the effect of surface tension on water waves at an inertial surface

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(Received 27 July 1982)

This note is to show that in the presence of surface tension small progressive waves can always exist in water having an inertial surface composed of uniformly distributed floating particles, in contrast to the known result in the absence of surface tension that precludes propagation under a surface that is too heavy.

It has been known for some time that infinitesimal time-harmonic progressive *gravity* waves of given angular frequency cannot propagate along the surface of an ideal liquid covered by a thin uniform distribution of non-interacting floating matter (broken ice, unstretched mat) if the layer, or 'inertial' surface, is too heavy. This differs from the result for a sufficiently light layer, when progressive waves are possible as for a free surface. Thus Peters (1950) and Weitz & Keller (1950) found that incident waves on a free surface might not be able to penetrate into an adjoining inertial surface for water of infinite and finite constant depth respectively. The purpose of this note is to remark on the interesting but apparently hitherto unnoticed fact that *capillary–gravity* waves can always propagate along any such surface layer without restriction; thus the presence of surface tension, however little, is enough to ensure propagation. (The layer may also be thought of here as a stretched heavy membrane.)

The problem is formulated for small irrotational motion under the effects of both gravity g and surface tension T of an ideal liquid (water) of volume density ρ which has an inertial surface composed of a thin uniform distribution of disconnected heavy floating matter of area density $\rho\epsilon$, say. The special case of a free surface corresponds to $\epsilon = 0$; and also in the absence of surface tension $T = 0$. Further, the motion is two-dimensional and time-harmonic of angular frequency σ, so may be described by a velocity potential of the form $\mathrm{Re}\,[\phi(x, y)\,e^{-i\sigma t}]$ at time t, where ϕ is complex-valued and depends on Cartesian coordinates x measured horizontally along the equilibrium inertial surface and y measured vertically downwards into the liquid from that surface. The inertial surface has depression $\mathrm{Re}\,[\eta(x)\,e^{-i\sigma t}]$ from the equilibrium position, where η is also complex-valued. First the case of *infinite* depth is taken.

The basic linearized requirements to be satisfied by ϕ are that it is the solution of Laplace's equation

$$\nabla^2\phi = 0 \quad \text{in} \quad y > 0$$

(continuity of mass in fluid region), subject to boundary conditions on $y = 0$ and as $y \to \infty$. The joint conditions relating ϕ, η at the inertial surface are the kinematic condition

$$\phi_y = -i\sigma\eta \quad \text{on} \quad y = 0$$

(inertial-surface particles remain there) and the dynamic condition

$$-i\sigma\phi = g\eta + (-i\sigma)^2 \epsilon\eta - \frac{T}{\rho}\eta_{xx} \quad \text{on} \quad y = 0$$

(equation of motion of inertial surface), in which the factor $-i\sigma$ corresponds to a time derivative; elimination of η in these then gives the single *inertial-surface condition*

$$K\phi + (1 - K\epsilon)\phi_y + M\phi_{yyy} = 0 \quad \text{on} \quad y = 0,$$

where $K = \sigma^2/g$ and $M = T/\rho g$ are positive constants (the latter may be zero). The usual form is recovered by putting $\epsilon = 0$ (no layer), as also is that of Peters (1950) by putting $M = 0$ (no surface tension). The remaining boundary condition is

$$|\nabla\phi| \to 0 \quad \text{as} \quad y \to \infty$$

(no motion at infinite depth).

Now, for $0 \leqslant K\epsilon < 1$ the inertial-surface condition has the form

$$K^*\phi + \phi_y + M^*\phi_{yyy} = 0 \quad \text{on} \quad y = 0,$$

where $K^* = K/(1-K\epsilon)$ and $M^* = M/(1-K\epsilon)$ are positive constants (the latter may be zero), so is merely a modification of the usual free-surface condition for $\epsilon = 0$, which certainly allows progressive waves. Thus the result for $M = 0$ may be extended. However, for $K\epsilon \geqslant 1$ the form is different and it might be expected that now no progressive waves are possible, as for $M = 0$ again. The results for $M = 0$ are those noted by Peters (1950) for an unstretched floating mat.

If in general (i.e. for any $K\epsilon \geqslant 0$) we look for a propagating solution satisfying the basic requirements in the form $\phi = e^{-ky \pm ikx}$ we find from the inertial-surface condition that remains to be satisfied that the prospective wavenumber $k > 0$ must satisfy the cubic equation

$$k(1 - K\epsilon + Mk^2) - K = 0,$$

which is therefore required to have a positive root; this is decided on by putting the equation in the form $1 - K\epsilon + Mk^2 = K/k$ and considering the intersection of the graphs of the two functions on either side (parabola, hyperbola): it is then evident that, *provided* $M > 0$, these intersect at a point in $k > 0$ for all values of $K\epsilon \geqslant 0$, so there is indeed a positive root. Thus for infinite depth *progressive waves can always exist under an inertial surface in the presence of surface tension.* This uniform result is exemplified for a wave source under an inertial surface in Rhodes-Robinson (1983); so also are the non-uniform results for $M = 0$, when the wave number is $k = K^*$ for $0 \leqslant K\epsilon < 1$ only.

Note that the phase speed c of the waves is given by

$$c^2 = \frac{\sigma^2}{k^2} = \frac{gK}{k^2} = \frac{g(1 + Mk^2)}{k(1 + k\epsilon)}$$

in terms of the wavenumber k. Also, if we consider the limit $M \to 0$, then for $0 \leqslant K\epsilon < 1$ it is seen that $k \to K/(1 - K\epsilon)$ so $c^2 \to g(1 - K\epsilon)^2/K$; whereas for $K\epsilon \geqslant 1$ it is found instead that $k \to \infty$ in such a way that $Mk^2 \to K\epsilon - 1$ so $c^2 \to 0$. These two limits depend continuously on $K\epsilon$ and correspond to the values for $M = 0$, when in the latter case there are no waves of course.

We next note that the above result for infinite depth holds also for *finite* constant depth h. The basic requirements on ϕ are now

$$\nabla^2\phi = 0 \quad \text{in} \quad 0 < y < h,$$

$$K\phi + (1-K\epsilon)\phi_y + M\phi_{yyy} = 0 \quad \text{on} \quad y = 0,$$

$$\phi_y = 0 \quad \text{on} \quad y = h$$

(motion horizontal on bottom), and it follows again that progressive waves having the form $\phi = \cosh k(h-y)\, e^{\pm ikx}$ can always propagate under an inertial surface for $M > 0$; for these

$$k(1-K\epsilon+Mk^2)\tanh kh - K = 0,$$

and

$$c^2 = \frac{g(1+Mk^2)\tanh kh}{k(1+k\epsilon \tanh kh)}.$$

For $M = 0$ there are again progressive waves for $0 \leqslant K\epsilon < 1$ and none for $K\epsilon \geqslant 1$, as noted by Weitz & Keller (1950) for floating ice.

Moreover, these results for a single liquid with an inertial surface hold also for two *superposed* liquids of either infinite or equal finite constant depth and height (the latter having a horizontal bottom and lid) that are separated by an inertial interface, since each of these problems may be reduced essentially to one for a corresponding single liquid with an inertial surface exactly as for the free-surface problems in Rhodes-Robinson (1980). If the interfacial tension is T, the volume densities of the lower and upper liquids $\rho, \rho' (0 < \rho' < \rho)$, and the area density of the inertial interface $(\rho-\rho')\epsilon$, say, then now

$$k(1-K\epsilon+M'k^2) - K' = 0$$

for infinite depth and height and

$$k(1-K\epsilon+M'k^2)\tanh kh - K' = 0$$

for equal finite depth and height h, where $K' = K(\rho+\rho')/(\rho-\rho')$ and $M' = T/(\rho-\rho')g$. For $M' = 0$ (no interfacial tension) there are progressive waves for $0 \leqslant K\epsilon < 1$ only.

In conclusion, we note that generalizations to three-dimensional motion can be made, involving for example axisymmetric cylindrical waves.

This work was done with the partial support of the Office of Naval Research, the Air Force Office of Scientific Research, the Army Research Office and the National Science Foundation while visiting the Applied Mathematics group at Stanford University, California. I would particularly like to thank Professors Joseph B. Keller and Harold Levine for their considerable interest. I am also grateful to Victoria University of Wellington for this period of leave.

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